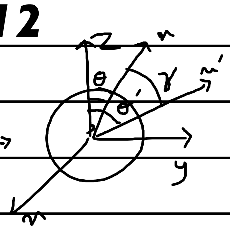
9/22/12

تابع گرین برای کره $\longrightarrow$ 

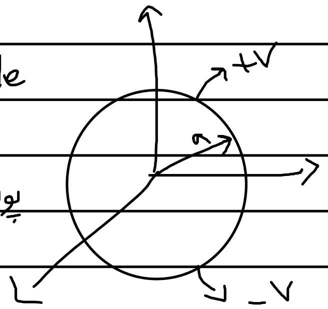
$$G(\vec{r},\vec{r}')=\frac{1}{|\vec{r}-\vec{r}'|}-\frac{a}{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r}'\right|}$$

$$G(\vec{r},\vec{r}')=\frac{1}{\sqrt{r^2+r'^2-2rr'\cdot\cos\gamma}}-\frac{1}{\sqrt{\frac{r^2r'^2}{a^2}+a^2-2rr'\cdot\cos\gamma}}$$

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$$

example

پوسته کروی



$$\Phi(\vec{r}')=\begin{cases}+V & 0<\theta<\frac{\pi}{2}\\ -V & \frac{\pi}{2}<\theta<\pi\end{cases}$$

$$r<a \Rightarrow \Phi(r)=?$$

$$\Phi(\vec{r})=-\frac{1}{4\pi}\oint_S \Phi(\vec{r}')\frac{\partial}{\partial n'}G(\vec{r},\vec{r}')\,da'$$

$$=\frac{-1}{4\pi}\oint_S \Phi(\vec{r}')\frac{\partial}{\partial n'}G(\vec{r},\vec{r}')\,a^2\sin\theta'\,d\theta'\,d\phi'$$

$$\frac{\partial}{\partial n'}G(\vec{r},\vec{r}') \longrightarrow \frac{r^2-a^2}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}$$

$$\Longrightarrow \Phi(\vec{r})=\frac{-1}{4\pi}\left[\int_0^{2\pi}\int_0^{\pi/2}V\cdot\frac{a^2\sin\theta'\,d\theta'\,d\phi'\cdot(r^2-a^2)}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}+\right.$$

$$\left.\int_0^{2\pi}\int_0^{\pi/2}\frac{-V\,a^2\sin\theta'\,d\theta'\,d\phi'(r^2-a^2)}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}\right]$$

$$\Longrightarrow \Phi(\vec{r})=\frac{-a(r^2-a^2)}{4\pi}\left[\int_0^{2\pi}\int_0^{\pi/2}\frac{V\sin\theta'\,d\theta'\,d\phi'}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\right.$$

$$\left.-\int_0^{2\pi}\int_0^{\pi/2}\frac{V\sin\theta'\cdot d\theta'\cdot d\phi'}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\right]$$

$$\Phi(\vec{x})=\frac{-aV(z^2-a^2)}{2}\left[\int_0^{\pi/2}\frac{\sin\theta'}{(z^2+a^2-2az\cos\theta')^{3/2}}-\int_{\pi/2}^{\pi}\frac{\sin\theta'\,d\theta'}{(z^2+a^2-2az\cos\theta')^{3/2}}\right]$$

$$\Longrightarrow \Phi(\vec{r}) = \frac{a\cdot V(z^2-a^2)}{2}\left(\frac{-1}{a^2}\right)\left[\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{a-z} - \frac{1}{a+z} + \frac{1}{\sqrt{z^2+a^2}}\right]$$

$$\Longrightarrow \Phi(\vec{r}) = \frac{V}{z}\left(a - \frac{a^2-z^2}{\sqrt{z^2+a^2}}\right) \quad z<a$$

برای $z>a$ $\longrightarrow$ $\Phi(\vec r)\frac{\partial}{\partial n'}G(\vec r,\vec r')\cdot da' \Longrightarrow \Phi(\vec r) = \frac{z^2-a^2}{4\pi a}\left[\int_0^{2\pi}\int_0^{\pi/2}\frac{V\sin\theta'\,d\theta'\,d\varphi'\,a^2}{(z^2+a^2-2az\cos\theta')^{3/2}} + \int_0^{2\pi}\int_{\pi/2}^{\pi}\frac{-V\sin\theta'\,d\theta'\,d\varphi'\,a^2}{(z^2+a^2-2az\cos\theta)^{3/2}}\right]$

$$-\frac{\partial}{\partial n'}G(\vec r,\vec r') = \frac{-(r^2-a^2)}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}$$

$$\Longrightarrow \Phi(\vec r) = \frac{aV(z^2-a^2)}{2}\left(\frac{-1}{az}\right)\left(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{z-a} - \frac{1}{z+a} + \frac{1}{\sqrt{z^2+a^2}}\right)$$

$$\Longrightarrow \Phi(\vec r) = V\left(1 - \frac{z^2-a^2}{z\sqrt{z^2+a^2}}\right)$$

محاسبه تابع گرین در دو بعد:

$$\nabla^2\Phi(\vec r) = -\rho(\vec r)/\varepsilon_0$$

$$\nabla^2 G(\vec r,\vec r') = -2\pi\,\delta(\vec r-\vec r')$$

انتقال دستگاه مختصات بر اساس $\vec r'=0$ $\longrightarrow \nabla^2 G(\vec r) = -2\pi\delta(\vec r) = -2\pi\delta(x)\delta(y) \Longrightarrow \nabla^2 G(r) = -2\pi\delta(r)\frac{1}{|J|} \Longrightarrow \nabla^2 G(r) = -2\pi\frac{1}{2\pi r}\delta(r) \Longrightarrow$

$r \to$ قطبی

$$|J| = \int_0^{2\pi}|J|\,d\varphi = \int_0^{2\pi}R\,d\varphi = 2\pi r$$

$$\nabla^2 G(r) = \frac{-1}{r}\delta(r) \Longrightarrow \frac{1}{r}\frac{d}{dr}\left(r\cdot\frac{dG(r)}{dr}\right) = \frac{-1}{r}\delta(r)$$

$(r\neq 0) \Longrightarrow \frac{d}{dr}\left(r\frac{dG(r)}{dr}\right) = 0 \Longrightarrow r\frac{dG(r)}{dr} = C_1 \Longrightarrow \frac{dG(r)}{dr} = \frac{C_1}{r} \Longrightarrow G(r) = C_1\ln r + C_2$

$(r'=0)$

$$\nabla^2 G(r) = \frac{-1}{r}\delta(r) \Longrightarrow \int_V \nabla^2 G(r)\cdot d^3\vec r = \int_V \frac{-1}{r}\delta(r)\cdot d^3\vec r$$

$$\Longrightarrow \int_V \vec\nabla\cdot\vec\nabla G(r)\,d^3\vec r = -\int_V\frac{1}{r}\delta(r)\cdot d^3\vec r \Longrightarrow \oint_S \vec\nabla G(r)\cdot\hat n\cdot da = -\int_V\frac{1}{r}\delta(r)\cdot d^3\vec r \Longrightarrow$$

$$\oint_S \frac{C_1}{r}\hat a_r\cdot r\,d\varphi\,dz\,\hat a_z = -\int\frac{1}{r}\delta(r)\,r\,dr\,d\varphi\,dz$$

$$\Longrightarrow C_1\int_0^{2\pi}d\varphi\int_0^h dz = -2\pi h \Longrightarrow C_1 2\pi h = -2\pi h \Longrightarrow \boxed{C_1=-1}$$

$\longrightarrow G(r) = -\ln r + C_2$

$G(r)\big|_{r=R} = 0 \Longrightarrow -\ln R + C_2 = 0$

$\Longrightarrow C_2 = \ln R$

$$\Longrightarrow G(r) = -\ln r + \ln R$$

$$\nabla^2 G(\vec{r}) = -2\pi\,\delta(\vec{r})$$

$$\nabla^2 \Phi(\vec{r}) = -\frac{\rho(\vec{r}')}{\varepsilon_0}$$

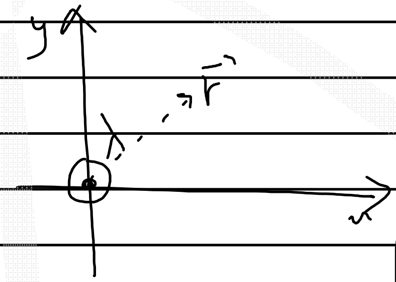
$$\vec{r}' = 0 \Longrightarrow \nabla^2 G(\vec{r}) = -2\pi\,\delta(\vec{r}) \Longrightarrow G(\vec{r}) = -\ln|\vec{r}| + \ln R$$

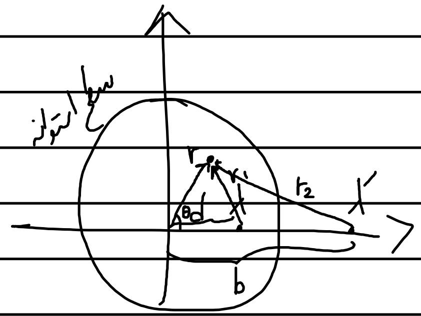
$$\vec{r}' \neq 0 \Longrightarrow G(\vec{r}, \vec{r}') = -\ln|\vec{r} - \vec{r}'|$$

تابع گرین برای دو بعد

بدون سطح مرزی

- محاسبه تابع گرین برای یک دایره در صفحه xoy





example

$$\varepsilon_0 E\, 2\pi r l = \lambda l$$

$$\Longrightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r} \Longrightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{a}_r$$

$$\Phi(\vec{r}) = -\int_{\substack{Ref \\ R_0}}^{r} \vec{E}\cdot d\vec{l} = \int_{R_0}^{r} \frac{\lambda}{2\pi\varepsilon_0 r}\hat{a}_r \cdot dr\,\hat{a}_r$$

$$\Longrightarrow \Phi(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{R_0}{r} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{R_0} \xrightarrow{\lambda \to 2\pi\varepsilon_0} G(\vec{r}, \vec{r}') = -\ln\frac{r}{R_0}$$

$$\vec{E}_1 = \frac{\lambda}{2\pi\varepsilon_0 R}\hat{a}_R \Longrightarrow \Phi_1 = -\int \vec{E}_1\cdot d\vec{L} = -\int_{R_0}^{R_1}\frac{\lambda}{2\pi\varepsilon_0 R}\hat{a}_R\cdot dR = \frac{-\lambda}{2\pi\varepsilon_0}\ln\frac{r_1}{R_0}$$

$$\vec{E}_2 = \frac{\lambda'}{2\pi\varepsilon_0 R'}\hat{a}_{R'} \Longrightarrow \Phi_2 = -\int E_2\cdot d\vec{L} = -\int_{R_0'}^{R_2}\frac{\lambda'}{2\pi\varepsilon_0 R'}dR' = \frac{-\lambda'}{2\pi\varepsilon_0}\ln\frac{r_2}{R_0'}$$

$$\Longrightarrow \Phi = \Phi_1 + \Phi_2 = \frac{-\lambda}{2\pi\varepsilon_0}\ln r_1 + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 - \frac{\lambda'}{2\pi\varepsilon_0}\ln r_2 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R_0'$$

$$\Phi = \frac{-\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+d^2-2rd\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\sqrt{R^2+b^2-2rb\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R_0'$$

اعمال شرایط دیریکله و اتصال محیط

$$\Phi(\vec{r})\Big|_{|\vec{r}|=a} = 0 \Longrightarrow \frac{-\lambda}{2\pi\varepsilon_0}\ln\sqrt{a^2+d^2-2ad\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\sqrt{a^2+b^2-2ab\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R_0' = 0$$

$$\Longrightarrow \frac{-\lambda}{2\pi\varepsilon_0}\ln\sqrt{a^2+d^2-2ad\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\frac{a}{d}\sqrt{d^2+\frac{b^2d^2}{a^2}-2\frac{bd^2}{a}\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R_0' = 0 \Longrightarrow$$

$$a^2 = \frac{b^2 d^2}{a^2} \Rightarrow b^2 = \frac{a^4}{d^2} \Rightarrow b = \frac{a^2}{d}$$

$$ad = \frac{bd^2}{a} \rightarrow b = \frac{a^2}{d}$$

$$\Longrightarrow -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{a^2+d^2-2ad\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\frac{a}{d} - \frac{\lambda'}{2\pi\varepsilon_0}\sqrt{a^2+d^2-2ad\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R_0' = 0$$

$$\downarrow$$

$$-\frac{1}{2\pi\varepsilon_0}\ln\sqrt{a^2+d^2-2ad\cos\theta}\,(\lambda+\lambda') = 0 \qquad \boxed{\begin{matrix}\lambda+\lambda'=0\\ \lambda'=-\lambda\end{matrix}}\,!$$

$$\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{a}{d} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 - \frac{\lambda}{2\pi\varepsilon_0}\ln R_0' = 0$$

$$\ln\left(\frac{a}{d}\cdot R_0\frac{1}{R_0'}\right) = 0 \Rightarrow \ln\frac{aR_0}{dR_0'} = 0 \Rightarrow \frac{aR_0}{dR_0'} = 1 \Rightarrow R_0' = \frac{aR_0}{d}$$

$$\Phi = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+d^2-2rd\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\sqrt{r^2+b^2-2rb\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R_0'$$

$$\Phi(\vec r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+d^2-2rd\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{R_0}{R_0'} \Longrightarrow$$

$$\Rightarrow \Phi(\vec r) = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{\sqrt{r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta}}{\sqrt{r^2+d^2-2rd\cos\theta}} + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{d}{a} \Longrightarrow G(\vec r,\vec r') = \Phi(\vec r)\Big|_{\lambda\to 2\pi\varepsilon_0} =$$

$$\ln\left(\frac{d\sqrt{r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta}}{a\sqrt{r^2+d^2-2rd\cos\theta}}\right)$$

$$G(\vec r,\vec r') = \ln\left(\frac{d\sqrt{r^2+b^2-2rb\cos\theta}}{a\sqrt{r^2+d^2-2rd\cos\theta}}\right) \Longrightarrow G(\vec r,\vec r') = \ln\left(\frac{r'}{a}\,\frac{|\vec r-\vec b|}{|\vec r-\vec d|}\right) = \ln\left(\frac{r'}{a}\,\frac{|\vec r-\frac{a^2}{r'^2}\vec r'|}{|\vec r-\vec r'|}\right)$$

روش (دوم) $\longmapsto G(\vec r,\vec r') = -\ln|\vec r-\vec r'| + \ln|\vec r-\vec r''| + C \longrightarrow \vec r'' = \frac{a^2}{r'^2}\vec r'$

$$G(\vec r,\vec r') = \ln\frac{|\vec r-\frac{a^2}{r'^2}\vec r'|}{|\vec r-\vec r'|} + C = \ln\sqrt{\frac{r^2+\frac{a^4}{r'^4}r'^2-\frac{2a^2}{r'^2}r\cos(\theta-\theta')}{r^2+r'^2-2rr'\cos(\theta-\theta')}} + C$$

$$G(\vec r,\vec r')\Big|_{r=a} = 0 \Longrightarrow \ln\frac{a}{r'}\sqrt{a} + C \longrightarrow C = -\ln\frac{a}{r'}$$

$$G(\vec r,\vec r') = \ln\left(\frac{r'\,|\vec r-\frac{a^2}{r'^2}\vec r'|}{a\,|\vec r-\vec r'|}\right)$$

$y''-y'-2y=2x^2-3$ Homework حل تمرین

$y(-1)=y(0)=0$

$$\Rightarrow G(x,s)\begin{cases} C_1e^x+C_2e^{-2x} & x<s\\ C_3e^x+C_4e^{-2x} & x>s\end{cases}$$

$G''(x,s)+G'(x,s)-2G(x,s)=\delta(x-s)$

$G''+G'-2G=0\ (s\neq x)$

$$\longrightarrow G(-1,s)=0 \longrightarrow C_1e^{-1}+C_2e^{2}=0$$

$$G(0,s)=0\longrightarrow C_3+C_4=0$$

$$G(x,s)\big|_{s^+}=G(x,s)\big|_{s^-}\longrightarrow C_1e^s+C_2e^{-2s}=C_3e^s+C_4e^{-2s}$$

$$G'(x,s)\big|_{s^+}-G'(x,s)\big|_{s^-}=1\longrightarrow$$

$$C_3e^s-2C_4e^{-2s}-(C_1e^s-2C_2e^{-2s})=1$$

$$y(x)=\int_{-1}^{0}G(x,s)f(s)ds=\int_{-1}^{x}(C_3e^x+C_4e^{-2x})(2s^2-3)\,ds+\int_{x}^{0}(C_1e^x+C_2e^{-2x})(2s^2-3)\,ds$$

Homework

$$\begin{cases}y''=f(x)\\ y(0)=1\\ y(1)=2\end{cases}$$

تابع گرین را بدست آورید
سپس $y(x)$ را بدست آورید!

$G''=\delta(x,s)$

$$G''=0\ (x\neq s)\Rightarrow\begin{cases}G=Ax+C & x>s\\ G=Bx+D & x<s\end{cases}$$

$\cancel{L[y]=f}$ $\cancel{\int f(s)[G(x,s)]ds}$

$\cancel{L[G]=f}$ $\cancel{=y''}$

تابع گرین برای شرط مرزی همگن صادق است فقط [تابع گرین دیریکله]

$\Longrightarrow$ تبدیل به شرایط مرزی همگن $\longrightarrow$ $y=u+v$, $y''=f(x)$

$$\begin{cases}u''+v''=f(x)\\ u(0)+v(0)=1\\ u(1)+v(1)=2\end{cases}\Rightarrow\begin{cases}u''=f(x)\\ u(0)=0\\ u(1)=0\end{cases}\ (1)\ \&\ \begin{cases}v''=0\\ v(0)=1\\ v(1)=2\end{cases}\ (2)$$

$$1 \longrightarrow G(x,s)=\begin{cases}(s-1)x & x<s\\ (s)(x-1) & x>s\end{cases}$$

$$2 \longrightarrow V = C_1 x + C_2$$

$$C_2 = 1$$
$$C_1 + C_2 = 2 \longrightarrow C_1 = 1$$

$$\longrightarrow V(x) = x+1$$

$$y(x) = \int_0^x s(x-1) f(s)\, ds + \int_x^1 (s-1)x f(s)\, ds + \overbrace{(x+1)}^{V(x)}$$

$$\Longrightarrow y(x) = (x-1)\int_0^x s\, f(s)\, ds + x\int_x^1 (s-1) f(s)\, ds + (x+1)$$

توابع orthogonal

$U_n(\xi)$ $(n = 1,2,3,\dots)$ orthonormal

در فاصله (a,b) راست به هنجار هستند

اگر توابع U_n به صورت مجذوری در این فاصله انتگرال پذیر باشند. و رابطه زیر برقرار باشد

$$\int_a^b U_n^*(\xi)\, U_m(\xi)\, d\xi = \delta_{m,n}$$

voice ...

if $f(\xi)$ انتگرال پذیر مجذوری در این فاصله $\longrightarrow f(\xi) \simeq \sum_{m=1}^{N} A_m U_m(\xi) \Longrightarrow A = \int_a^b \left| f(\xi) - \sum_{m=1}^{N} A_m U_m(\xi) \right|^2 d\xi$

$$A_{min} \Longrightarrow \boxed{\text{مشتق صفر}}\ 0$$

$$A_m = \int_a^b f(\xi') U_m^*(\xi')\, d\xi'$$

$$\longrightarrow f(\xi) = \sum_{m=1}^{N} A_m U_m(\xi) \quad \boxed{00}$$

voice

اگر $U_n(\xi)$ ها مجموعه‌ای کامل باشد $\longrightarrow f(\xi) = \sum_{m=1}^{\infty} A_m U_m(\xi)$

به میانگین $f(\xi)$ میل می‌کند

$$f(\xi) = \sum_{m=1}^{\infty} A_m U_m(\xi) \Longrightarrow f(\xi) = \sum_{m=1}^{\infty} \int_a^b f(\xi') U_m^*(\xi')\, d\xi'\, U_m(\xi) \longrightarrow$$

$$f(\xi)=\int_a^b \sum_{m=1}^{\infty} U_m^*(\xi')\,U_m(\xi)\,f(\xi')\,d\xi' \quad \boxed{\text{oo}}$$

voice

$$\delta(\xi-\xi')$$

$$\Rightarrow \boxed{\sum_{m=1}^{\infty} U_m^*(\xi')\,U_m(\xi)=\delta(\xi-\xi')}$$

شرط انتگرال پذیر مجذوری بودن $f(\xi)$ $\longrightarrow \boxed{\text{oo}}$ $\left(-\frac{a}{2},\frac{a}{2}\right)$ $U_m=\frac{1}{\sqrt{a}}\,e^{\frac{i(2\pi m x)}{a}}$; $m=0,\pm1,\pm2$

$$f(x)=\sum_m A_m U_m(x) \Rightarrow f(x)=\sum_m A_m \frac{1}{\sqrt{a}}\,e^{\frac{i(2\pi m x)}{a}}$$

$$A_m=\frac{1}{\sqrt{a}}\int_{-a/2}^{a/2} f(x')\,e^{-i\frac{(2\pi m x')}{a}}\,dx'$$

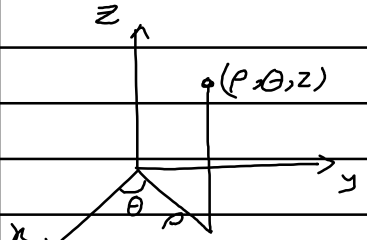
$$\left.\begin{array}{l} a\to\infty \\ \frac{2\pi m}{a}=K \end{array}\right| \to \sum_m \xrightarrow{\text{تبدیل}} \int_{-\infty}^{+\infty} dm \Longrightarrow dm=\frac{a}{2\pi}\,dK$$

$$f(x)=\int_{-\infty}^{+\infty}\frac{a\,dk}{2\pi}\,\frac{1}{\sqrt{a}}\,e^{ikx}\sqrt{\frac{2\pi}{a}}\,dK \Longrightarrow f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ikx}A(k)\,dK$$

!!

نگاشت همدیس Conformal mapping [تابع باید در هر نقطه وجود داشته باشد] $\leftarrow$ $z=x+iy$

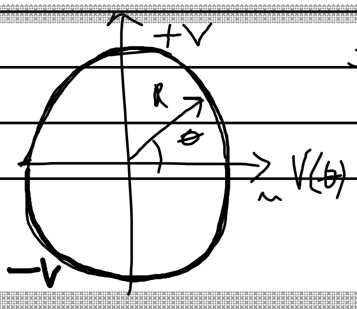
$f'(z)=$ وجود دارد



متقارن به حسب z $\Rightarrow \Phi=\Phi(\rho,\theta)$

$$\nabla^2\Phi(\rho,\theta)=0 \Longrightarrow \Phi(\rho,\theta)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}\rho(a_n\sin n\theta+b_n\cos n\theta)+\rho^{-n}(c_n\sin n\theta+D_n\cos n\theta)$$

مسأله ـ یک پوسته استوانه‌ای طویل به شعاع R دارای پتانسیل به صورت زیر است



$$V(\theta)=\begin{cases} +V & 0\le\theta<\pi \\ -V & \pi\le\theta<2\pi \end{cases}$$

محاسبه پتانسیل در نقاط داخل استوانه

حل → پتانسیل الکترواستاتیکی در داخل استوانه (ρ,φ) $(\rho<R)$

$$\Phi(\rho,\varphi)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}\left[\rho^{n}(a_n\cos n\varphi+b_n\sin n\varphi)+\rho^{-n}(C_n\cos n\varphi+D_n\sin n\varphi)\right]$$

$\Phi\big|_{\rho=0}$ باید مقدار داشته باشد $\Rightarrow b_0=0$, $C_n=d_n=0$

$$\Phi(\rho,\varphi)=a_0+\sum_{n=1}^{\infty}\rho^{n}(a_n\cos n\varphi+b_n\sin n\varphi)$$

$$\Phi(R,\varphi)=f(\varphi)=a_0+\sum_{n=1}^{\infty}R^{n}(a_n\cos n\varphi+b_n\sin n\varphi)\xrightarrow{\int_0^{2\pi}}\int_0^{2\pi}f(\varphi)\,d\varphi=a_0 2\pi\longrightarrow\boxed{a_0=\frac{1}{2\pi}\int_0^{2\pi}f(\varphi)\,d\varphi}$$

$$\int_0^{2\pi}f(\varphi)\cos m\varphi\,d\varphi=a_0\underbrace{\int_0^{2\pi}\cos m\varphi\,d\varphi}_{0}+\sum_{n=1}^{\infty}R^{n}\left[a_n\underbrace{\int_0^{2\pi}\cos m\varphi\cos n\varphi\,d\varphi}_{0}+b_n\underbrace{\int_0^{2\pi}\cos m\varphi\sin n\varphi\,d\varphi}_{0}\right]\Rightarrow$$

$$\Rightarrow a_c\int_0^{2\pi}f(\varphi)\cos m\varphi\,d\varphi=R^{m}a_m\pi\longrightarrow$$

$$a_n=\frac{1}{R^{n}\pi}\int_0^{2\pi}f(\varphi)\cos n\varphi\,d\varphi$$

$$b_n=\frac{1}{R^{n}\pi}\int_0^{2\pi}f(\varphi)\sin n\varphi\,d\varphi$$

$$a_0=\frac{1}{2\pi}\left[\int_0^{\pi}V\,d\varphi+\int_{\pi}^{2\pi}(-V)\,d\varphi\right]=0$$

$$a_n=\frac{1}{\pi R^{n}}\left[\int_0^{\pi}V\cos n\varphi\,d\varphi+\int_{\pi}^{2\pi}(-V)\cos n\varphi\,d\varphi\right]=0$$

$$b_n=\frac{1}{\pi R^{n}}\left[\int_0^{\pi}V\sin n\varphi\,d\varphi+\int_{\pi}^{2\pi}(-V)\sin n\varphi\,d\varphi\right]\Rightarrow b_n=\begin{cases}0 & n \text{ زوج}\\ \frac{4V}{\pi n R^{n}} & n \text{ فرد}\end{cases}$$

$$\Phi(\rho,\varphi)=\frac{4V}{\pi}\sum_{n=1,\ \text{فرد}}^{\infty}\rho^{n}\sin n\varphi\,\frac{1}{\pi R^{n}}$$

$(\rho e^{i\varphi})^{n}=z^{n}$

$$\Phi_1=\frac{4V}{\pi}\sum_{n=1,\ \text{فرد}}^{\infty}\rho^{n}e^{in\varphi}\cdot\frac{1}{n\cdot R^{n}}$$

$$\Phi=I_m(\Phi_1)\ ,\ \Phi_1=\frac{4V}{\pi}\sum_{n=1,\ \text{فرد}}^{\infty}\frac{1}{n}\left(\frac{z}{R}\right)^{n}\longrightarrow\frac{d\Phi_1}{dz}=\frac{4V}{\pi}\sum_{n=1,\ \text{فرد}}^{\infty}\frac{1}{n}\cdot n\cdot\frac{1}{R}\left(\frac{z}{R}\right)^{n-1}$$

$$\longrightarrow\frac{d\Phi_1}{dz}=\frac{4\pi}{\pi R}\sum_{n=1,\ \text{فرد}}^{\infty}\left(\frac{z}{R}\right)^{n-1}=\frac{4V}{\pi R}\left[1+\frac{z^2}{R^2}+\frac{z^4}{R^4}+\dots\right]=\frac{4V}{R\pi}\,\frac{1}{1-\frac{z^2}{R^2}}=\frac{4VR}{\pi}\,\frac{1}{R^2-z^2}\Rightarrow$$

$$\left|\frac{z^2}{R^2}\right|=\frac{\rho^2}{R^2}<1$$ همگرا!

$$\frac{d\Phi_1}{dz} = \frac{4VR}{\pi}\left(\frac{1}{2R}\right)\left(\frac{1}{R-z}+\frac{1}{R+z}\right) = \frac{2V}{\pi}\left(\frac{1}{z+R}-\frac{1}{z-R}\right)$$

$$\Longrightarrow \Phi_1 = \frac{2V}{\pi}\ln\frac{z+R}{z-R} \rightarrow \Phi_1 = \frac{2V}{\pi}\ln\frac{x+iy+R}{x+iy-R} \longrightarrow$$

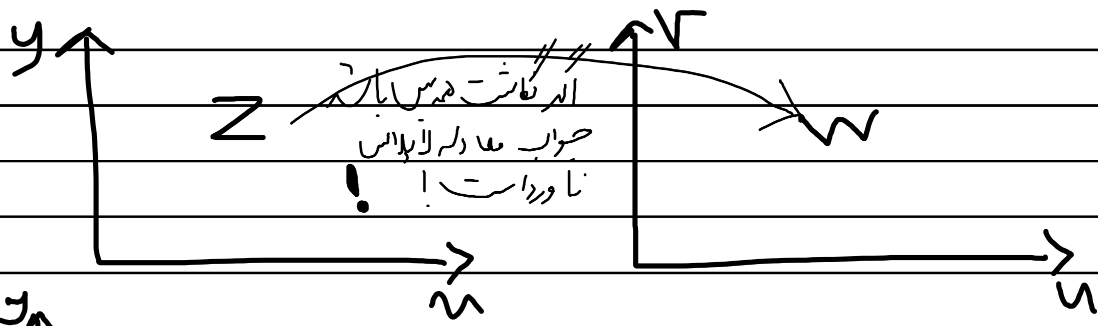
$$\Phi_1 = \frac{2V}{\pi}\ln\left(\frac{x+iy+R}{x+iy-R}\cdot\frac{x-R-iy}{x-R-iy}\right) = \frac{2V}{\pi}\ln\left[\frac{x^2+y^2-R^2-i2Ry}{(x-R)^2+y^2}\right] = \frac{2V}{\pi}\ln\left[\frac{\sqrt{(x^2+y^2-R^2)^2+4R^2y^2}}{(x-R)^2+y^2}\times e^{i\tan^{-1}\frac{-2Ry}{x^2+y^2-R^2}}\right]$$

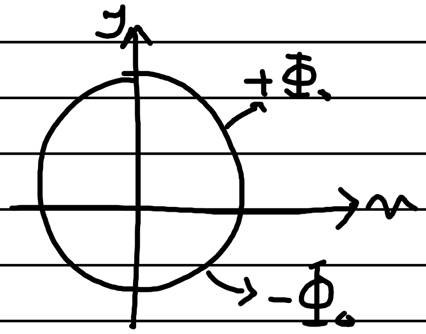
$$\Longrightarrow \Phi_1 = \frac{2V}{\pi}\left[\ln\frac{\sqrt{(x^2+y^2-R^2)^2+4R^2y^2}}{(x-R)^2+y^2} + i\tan^{-1}\frac{-2Ry}{x^2+y^2-R^2}\right]$$

$$\Longrightarrow \Phi = \mathrm{Im}(\Phi_1) = \frac{2V}{\pi}\tan^{-1}\frac{2Ry}{R^2-x^2-y^2} \qquad R=1 \rightarrow \Phi(x,y) = \frac{2V}{\pi}\tan^{-1}\frac{2y}{1-x^2-y^2} \rightarrow$$

$$\Phi(\rho,\varphi) = \frac{2V}{\pi}\tan^{-1}\frac{2\rho\sin\varphi}{1-\rho^2}$$

Conformal mapping ⟶





$z = e^{i\theta}$ OR $|z| = 1$

$$W(z) = \ln\frac{1+z}{1-z} = \ln(1+z) - \ln(1-z)$$

$$\frac{dW}{dz} = \frac{1}{1+z} - \frac{-1}{1-z} = \frac{1-z+1+z}{1-z^2} = \frac{2}{1-z^2}$$ در همه نقاط تحلیلی به جز

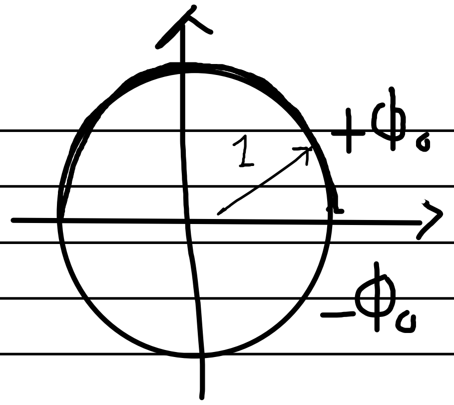
$1-z^2=0 \rightarrow z=\pm 1$

⟶ نگاشت، یک نگاشت همدیس است

$$W(z) = U+iV = \ln\frac{1+z}{1-z} = \frac{1+x+iy}{1-x-iy} = \ln\left[\frac{1+x+iy}{1-x-iy}\cdot\frac{1-x+iy}{1-x+iy}\right] = \ln\left[\frac{1-x^2-y^2+2iy}{1-2x+x^2+y^2}\right]$$

$$W(z) = \ln\left[\frac{\sqrt{(1-x^2-y^2)^2+4y^2}}{1-2x+x^2+y^2}\cdot e^{i\tan^{-1}\frac{2y}{1-x^2-y^2}}\right] \longrightarrow W(z) = U(x,y)+iV(x,y) = \underbrace{\ln\frac{\sqrt{(1-x^2-y^2)^2+4y^2}}{1-2x+x^2+y^2}}_{U(x,y)} + i\underbrace{\tan^{-1}\frac{2y}{1-x^2-y^2}}_{V(x,y)}$$

$$\begin{cases} u(x,y) = \ln \dfrac{\sqrt{(1-x^2-y^2)^2+4y^2}}{|1-2x+x^2+y^2|} \\[2ex] V(x,y) = \tan^{-1} \dfrac{2y}{1-x^2-y^2} \end{cases}$$



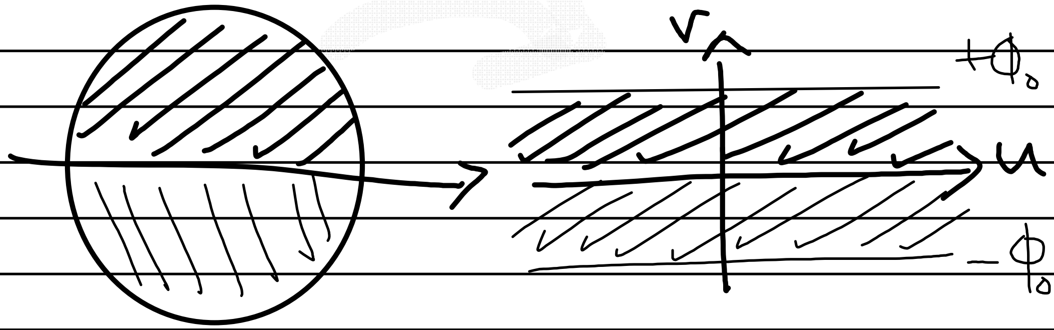
نیم صفحه $\rho = \sqrt{x^2+y^2} < 1$

نیمه بالای دایره $\begin{cases} x^2+y^2 = 1 \\ y > 0 \end{cases} \Longrightarrow u(x,y) = \ln \dfrac{2y}{2|1-x|} = \ln \dfrac{y}{|1-x|}$

$V(x,y) = \tan^{-1} \dfrac{2y}{1-x^2-y^2} = \tan^{-1} \infty = +\dfrac{\pi}{2}$

نیمه پایینی $\begin{cases} x^2+y^2 = 1 \\ y < 0 \end{cases} \longrightarrow u(x,y) = \ln \left| \dfrac{y}{1-x} \right|$

$V(x,y) = \tan^{-1}(-\infty) = -\dfrac{\pi}{2}$



در صفحه u,v: $\nabla^2 \Phi = 0 \longrightarrow \dfrac{d^2\Phi}{dV^2} = 0 \longrightarrow \Phi = C_1 V + C_2$

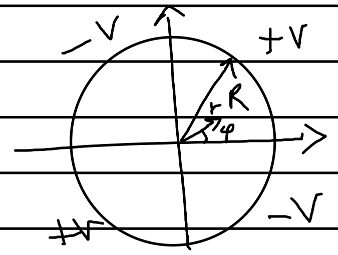
$\begin{cases} \Phi_0 = C_1 \frac{\pi}{2} + C_2 \\ -\Phi_0 = C_1 \left(-\frac{\pi}{2}\right) + C_2 \end{cases} \longrightarrow \begin{cases} C_1 = \frac{2\Phi_0}{\pi} \\ C_2 = 0 \end{cases}$

$\Phi = \dfrac{2\Phi_0}{\pi} \tan^{-1} \dfrac{2y}{1-x^2-y^2} \Longrightarrow$

$\Phi = \dfrac{2\Phi_0}{\pi} \tan^{-1} \dfrac{2\rho \sin\varphi}{1-\rho^2}$

Homework

استوانه رسانای طویل به شعاع R مطابق شکل زیر است. پیدا کنید پتانسیل در نقاط داخل



جواب $\longrightarrow \Phi(r,\varphi) = \dfrac{2V}{\pi} \tan^{-1} \dfrac{2R^2 r^2 \sin 2\varphi}{R^4 - r^4} \quad r < R$

— با استفاده از هماهنگ‌های استوانه‌ای

— 〃 〃 روش نگاشت همدیس